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Notes on Baby Rudin, Chapters 2–5

These are notes I took on Chapters 2–5 of Baby Rudin (aka Principles of Mathematical Analysis by Walter Rudin) while taking the course MATH 321-1 MENU: Real Analysis at Northwestern University in the fall of 2020. Each section of these notes corresponds to a chapter of Rudin, and sections themselves are split into subsections containing definitions, theorems, and proof techniques. (I may sometimes reference lecture numbers in these notes, which I did at the time because recordings were available due to the pandemic.)

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2 Basic Topology

2.1 Definitions

Definition 2.1 (Boundary) *The boundary of E is $\overline{E} \setminus E^o$, where $\overline{E} = E \cup E'$ is the closure of E, E' is the set of limit points of E, and E^o is the interior of E.*

See also 2.9.

Definition 2.2 (Open Relative To) *Say $Y \subset X$ where X is a metric space. Then Y is itself a metric space. $E \subset Y$ is open relative to Y if for all $x \in E$, there exists $r > 0$ such that if $y \in Y$ and $d(x, y) < r$, then $y \in E$.*

For example, $[0, 1/2)$ is open relative to $[0, 1]$ but not open relative to $\mathbb{R}$.

See also 2.13.

Definition 2.3 (Open Cover) *Let $E \subset X$ where X is a metric space. An open cover of E is a collection of open sets $G_\alpha \subset X$ such that $E \subset \bigcup_\alpha G_\alpha$. (No assumption is made about how many open sets are used. E.g., could be uncountable.)*

Definition 2.4 (Compact) *$K \subset X$ is compact if every open cover of K has a finite subcover. I.e., if $K \subset \bigcup_\alpha G_\alpha$, where G_α are open, then $K \subset \bigcup_{i=1}^n G_i$ for some n.*

See also 2.14, 2.15, 2.16, 2.17, 2.18, 2.19, 2.21.

Definition 2.5 (k-Cell) *A k-cell in $\mathbb{R}^k$ is a set of the form*

$$C = [a_1, b_1] \times [a_2, b_1] \times \cdots \times [a_k, b_k] = \left\{ x \in \mathbb{R}^k : x_j \in [a_j, b_j],\ j = 1, 2, \ldots, k \right\}.$$

The diameter of a k-cell is

$$\delta = \left(\sum_{j=1}^{k} (b_j - a_j)^2 \right)^{1/2}.$$

See 2.20.

Definition 2.6 (Separated, Connected)

(a) A, B are separated if $A \cap \overline{B} = \overline{B} \cap A = \emptyset$.

(b) E is connected if E is not the union of two nonempty separated sets. See also 2.22, 2.23.

Definition 2.7 (Perfect) *A set P is perfect if it is closed and every point of P is a limit point of P. See 2.24.*

2.2 Theorems

Theorem 2.8 (Open/Closed Complement) *A set is open if and only if its complement is closed. Furthermore, the entire metric space and empty set are both closed and open.*

Theorem 2.9 (Closure and Boundary)

(a) The closure of E is closed.

(b) E is closed if and only if $E = \overline{E}$.

(c) A point lies in the boundary of E if and only if it belongs to both $\overline{E}$ and $\overline{E^c}$.

Theorem 2.10 (Closure) *The closure of a set is the smallest closed set containing it.*

Theorem 2.11 (Union/Intersection of Closed/Open Sets)

(a) The union over an arbitrary (including uncountable) number of open sets is open.

(b) Intersection of any number of closed sets is closed.

(c) Intersection of finitely many open sets is open.

(d) Union of finitely many closed sets is closed.

Theorem 2.12 (De Morgan's laws)

(a) $(\bigcap_\alpha F_\alpha)^c = \bigcup_\alpha F_\alpha^c$

(b) $(\bigcup_\alpha F_\alpha)^c = \bigcap_\alpha F_\alpha^c$

Theorem 2.13 (Open Relative To) *If $Y \subset X$, then $E \subset Y$ is open relative to Y if and only if $E = Y \cap G$ for some $G \subset X$ where G is open.*

Theorem 2.14 (Preservation of Compactness) *If $K \subset Y \subset X$, then K is compact relative to Y if and only if K is compact relative to X.*

Theorem 2.15 (Closed Subsets in Compact Set) *If $K \subset X$ is compact, then every closed subset of K is compact.*

Theorem 2.16 (Compact Implies Closed/Bounded) *Every compact subset of a metric space is closed and bounded. (Note that the converse is only true in some metric spaces—see 2.19.)*

Theorem 2.17 (Compact, Closed Intersection) *If F is closed and K is compact, then $F \cap K$ is compact.*

Theorem 2.18 (Finite Intersection Property) *If $\{K_\alpha\}$ are nonempty, compact sets in X such that the intersection of any finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap_\alpha K_\alpha \neq \emptyset$. (I.e., the intersection of all of them is nonempty, even if there are uncountably many.)*

Theorem 2.19 (Heine-Borel Theorem) *A subset E of $\mathbb{R}^k$ is compact if and only if E is closed and bounded.*

Theorem 2.20 (Nested k-Cells) *A nested sequence $C_1 \supseteq C_2 \supseteq \dots$ of k-cells has $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$. (This is actually just an application of 2.18 since k-cells are nonempty and compact.)*

Theorem 2.21 (Equivalent Def. of Compactness) *E is compact if and only if every infinite subset of E has a limit point in E.*

Theorem 2.22 (Connected in $\mathbb{R}^1$) *$E \subset \mathbb{R}^1$ is connected if and only if for all $x, y \in E$ and z with $x < z < y$, we have $z \in E$.*

Theorem 2.23 (Properties of Connected Sets)

(a) If A is connected and $A \subseteq B \subseteq \overline{A}$, then B is also connected.

(b) The union of a collection of connected sets with some point in common is connected to all of them.

(c) If A, B are separated sets in X, $A \cup B = X$, and $C \subset X$ is connected, then $C \subset A$ or $C \subset B$.

(d) A is connected if and only if for every pair of open sets U, V such that $A \subset U \cup V$ and $A \cap V \cap A = \emptyset$, we have $A \subset U$ or $A \subset V$.

(e) If A is connected, then $\overline{A}$ is connected.

Theorem 2.24 (Perfect Sets Are Big) *If $P \subset \mathbb{R}^k$ is nonempty and perfect, then P is uncountable. (Perfect sets are big.)*

2.3 Proof techniques

Technique 2.25 *A method for obtaining a finite open cover of a compact set: Put a ball around every single point in the compact set and then cut down to a finite number of them. It is used at the end of Lecture 8 to prove 2.16.*

3 Numerical Sequences and Series

3.1 Definitions

Definition 3.1 (Converges) *Let X be a metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges if there exists $x \in X$ such that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $d(x_n, x) < \varepsilon$.*

We can write this as $x_n \stackrel{n\to\infty}{\longrightarrow} x$ *or* $\lim_{n\to\infty} x_n = x$. *Furthermore, if the sequence converges, the limit* x *is unique.*

Otherwise, we say $\{x_n\}$ *diverges.*

See 3.21.

Definition 3.2 (Range, Bounded Sequence) *The range of a sequence* $\{x_n\}$ *is the set* $\{x_n\}$. *(I.e., convert the sequence to a set. Ruding uses the bracket notation for both sequences and sets, which is what I'm using here too.) Can be finite or infinite.*

We say the sequence $\{x_n\}$ *is bounded if the range is bounded.*

Definition 3.3 (Subsequence) *Let* $\{x_n\}$ *be a sequence. A subsequence is* $\{x_{n_k}\}$. *You must keep an infinite number of things. (You can't just take, e.g., one number out of your sequence.) In general, we actually define sequences in this class to always be infinite! Note also that the things you take out do not have to be adjacent. E.g., you could take out all of the odd-indexed entries from your original sequence. You do have to keep everything in the same order as in the original sequence. A subsequence may or may not converge.*

See 3.20, 3.22.

Definition 3.4 (Cauchy Sequence) *A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is Cauchy if for all* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $d(x_n, x_m) < \varepsilon$ *for all* $m, n \geq N$.

What is the difference between a Cauchy sequence and a convergent sequence? Recall that $\{x_n\} \to x$ *if for all* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$, *we have* $d(x_n, x) < \varepsilon$. *The key difference then is to converge, you must actually converge to some* x *in your (metric) space. However, a sequence being Cauchy does not force you to converge to something in your space. It only talks about distances between points far along in your sequence. To sum things up: for a Cauchy sequence, there is no explicit limit. See 3.26 for an example of this.*

Definition 3.5 (Complete) *A metric space in which every Cauchy sequence converges is complete. E.g.,* $\mathbb{R}^k$ *is complete, but* $\mathbb{Q}$ *isn't.*

Definition 3.6 (Completion) *From Wolfram MathWorld: "A metric space* X *which is not complete has a Cauchy sequence which does not converge. The completion of* X *is obtained by adding the limits to the Cauchy sequences." I.e., it is the smallest superset of* X *that is complete.*

E.g., $\mathbb{R}$ *is the completion of* $\mathbb{Q}$. *(In fact, this can be taken as a definition of* $\mathbb{R}$.*)*

Definition 3.7 (Totally Bounded) *A space is totally bounded if it can be covered by finitely many subsets of fixed size (for any size).*

This is a more general version of compactness. Compactness requires that for any open cover, you can reduce down to a finite subcover. Totally bounded forces you to do the same, except you are only considering covers where all the balls are the same size. (So there are less covers you have to consider.)

Definition 3.8 (Monotonically Increasing/Decreasing)

(a) $\{x_n\}$ is monotonically increasing if $x_n \leq x_{n+1}$ for all n.

(b) $\{x_n\}$ is monotonically decreasing if $x_n \geq x_{n+1}$ for all n.

Note that constant sequences are both monotonically increasing and decreasing.

We say a sequence is monotonic if it is either monotonically increasing or decreasing.

Definition 3.9 (Diverges to ∞) *If x_n is a sequence of real numbers such that for all $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n > M$, then $x_n \to +\infty$. (It diverges to $+\infty$.) The case with $-\infty$ works similarly.*

Definition 3.10 (Subsequential Limits) *If $\{x_n\}$ is a sequence of reals, the set of all subsequential limits is the set of all possible values of x, where $x_{n_k} \to x$ (along the subsequence $\{x_{n_k}\}$). (I.e., the set of all values you can get by taking limits of subsequences.) This set can include $\pm\infty$. It can be finite or infinite but not empty.*

Definition 3.11 ($\limsup$ and $\liminf$) *Let E denote the set of subsequential limits (see 3.10) of some sequence $\{x_n\}$. We make the definitions*

$$\limsup_{n\to\infty} x_n = \overline{\lim_{n\to\infty}} x_n = \sup E,$$
$$\liminf_{n\to\infty} x_n = \underline{\lim_{n\to\infty}} x_n = \inf E.$$

If $\{x_n\}$ is bounded and $M = \limsup_{n\to\infty} x_n$, then for all $\varepsilon > 0$, there are finitely many terms in $\{x_n\}$ that are larger than $M + \varepsilon$. On the other hand, there are infinitely many terms greater than $M - \varepsilon$. This is really just an equivalent way of phrasing one direction of 3.33.

If $\{x_n\}$ converges, then $\limsup x_n = \liminf x_n = \lim x_n$.

Note: $\lim x_n$ may not exist, but $\limsup$ and $\liminf$ always exist (and may be $\pm\infty$, but this is allowed).

An equivalent definition of $\limsup$ is

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{m>n} x_m \right).$$

I.e., you are taking the supremum of the (infinite) set of tail values as you move the tail farther and farther down.

Definition 3.12 (Series and Convergence, Divergence) *In this class, we always take a series to mean an infinite series:*

$$\sum_{k=1}^{\infty} x_k.$$

The nth partial sum S_N is defined as

$$S_n = \sum_{k=1}^{n} x_k.$$

If $\{S_n\}$ converges to s, then the series $\sum_{k=1}^{\infty} x_k$ converges to s. If $\{S_n\}$ diverges, then the series diverges.

Definition 3.13 (Geometric Series) *If $0 \leq x < 1$, then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \geq 1$, the series diverges. (The limit of the terms don't go to 0, so this follows immediately from 3.41.)

Definition 3.14 (Absolute Convergence) *We say that $\sum a_n$ converges absolutely if $\sum |a_n|$ also converges.*

Note that the convergence of $\sum |a_n|$ implies the convergence of $\sum a_n$ since

$$\sum a_n \leq \left| \sum a_n \right| \leq \sum |a_n|$$

by the triangle inequality. So absolute convergence implies convergence. The converse is not true, as shown by the fact that the alternating harmonic series converges (see 3.53) whereas the harmonic series diverges.

Definition 3.15 (e) *Define e by*

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

This series converges because the partial sums are bounded and monotonic.

Definition 3.16 (Power Series) *If $\{c_n\}$ are complex numbers, then*

$$\sum_{n=1}^{\infty} c_n z^n$$

is a power series. The c_n are the coefficients.

The difference between a power series and a regular series is that in a power series, we have some variable. (Here z is our variable.)

As an example, we could set $\{c_n\}$ to just be the sequence of all 1's, which means that our power series is just $\sum_{n=1}^{\infty} z_n$. This converges if and only if $|z| < 1$; such z form a circle in the complex plane. This "circle of convergence" behavior is extremely common, where it converges inside the circle and diverges outside the circle. We have to be more careful about what happens on the circle.

Definition 3.17 (Radius of Convergence) *Given a power series $\sum c_n z_n$ where the $c_n \in \mathbb{C}$ are complex coefficients and z is a variable, we say R is the radius of convergence if the series converges for $|z| < R$ and diverges for $|z| > R$. (It doesn't matter what happens when $|z| = R$.)*

Definition 3.18 (Cauchy Product of Series) *Suppose we have two series $\sum a_n$ and $\sum b_n$. Define*

$$c_n := \sum_{k=0}^{n} a_k b_{n-k}.$$

Then $\sum c_n$ is called the Cauchy product of the two series. (In fact, the Cauchy product is taken to be the default product between series. So when we say "take the product of these two series," we really mean the Cauchy product.)

Where does this definition come from? The idea lies in multiplying two power series:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n &= a_0 b_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \dots \\ &= c_0 + c_1 z + c_2 z^2 + \dots\end{aligned}$$

where c_n is exactly how we defined it above. Then if you take z to be 1, you recover the Cauchy product of $\sum a_n$ and $\sum b_n$.

Definition 3.19 (Rearrangement) *A sequence $\{b_n\}$ is a rearrangement of $\{a_n\}$ if there is a bijective map $\varphi : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\varphi(n)}$. (Informally, you are just permuting the elements of the original sequence.)*

3.2 Theorems

Theorem 3.20 (Subsequence Convergence)

(a) If $\{x_n\}$ converges, so does any subsequence, and the limit is the same.

(b) If $\{x_n\}$ is a sequence in a compact metric space, then it has a convergent subsequence.

(c) A bounded sequence in $\mathbb{R}^k$ has a convergent subsequence. (We showed that a bounded set in $\mathbb{R}^k$ lies in a compact set.)

Theorem 3.21 (Equivalent Def. of Convergence) *x_n converges to x if and only if every neighborhood of x contains all but finitely many points of $\{x_n\}$.*

Theorem 3.22 (Convergent Implies Bounded) *Any convergent sequence is bounded.*

Theorem 3.23 (Closure, Convergence) *If $E \subset X$, then $x \in \overline{E}$ if and only if there exists a sequence $\{x_n\}$ in E with $x_n \to x$. (Rephrasing of E is closed if and only if it contains the limits of all convergent sequences in E.)*

Theorem 3.24 *If $\{x_n\}, \{y_n\}$ are sequences of complex numbers with $x_n \to x, y_n \to y$, then*

(a) $\lim x_n + y_n = x + y$

(b) $\lim cx_n = cx$, $\lim c + x_n = c + x$, *where $c \in \mathbb{C}$.*

(c) $\lim x_n y_n = xy$

(d) $\lim \frac{1}{x_n} = \frac{1}{x}$ *if $x_n \neq 0$ for all n and $x \neq 0$.*

(e) *If $x_n \in \mathbb{R}^k$ and $x_n = (x_{1,n}, \ldots, x_{k,n})$, then $x_n \to x = (x_1, \ldots, x_k)$ if and only if $\lim x_{j,n} = x_j$ for all $1 \leq j \leq k$.*

Theorem 3.25 (Cauchy Sequences Are Bounded) *Cauchy sequences are bounded. (This actually wasn't proven in lecture, but I thought I would include it anyway.)*

Theorem 3.26 (Convergent Implies Cauchy) *A convergent sequence is also Cauchy.*

The converse is not true. E.g., take the sequence of rationals that converges to $\sqrt{2}$. In $\mathbb{Q}$, this sequence is Cauchy. But it does not converge in $\mathbb{Q}$.

Theorem 3.27 (Convergent Equivalent to Cauchy in $\mathbb{R}^k$) *A sequence in $\mathbb{R}^k$ is convergent if and only if it is Cauchy. (Note that one direction follows immediately from 3.26.)*

Theorem 3.28 (Compact Metric Spaces Are Complete) *Compact metric spaces are complete. The converse if false. ($\mathbb{R}^k$ is complete but not compact.)*

Theorem 3.29 (Closed Subset of Complete Metric Space) *A closed subset of a complete metric space is itself complete.*

This follows by taking a Cauchy sequence in the closed subset, and noting that first of all that it converges to something in the parent space due to the parent spaces completeness. Then note that it must actually converge to something in the subset since the subset is closed.

Theorem 3.30 (Compact $\iff$ Complete and Totally Bounded) *A metric space is compact if and only if it is complete and totally bounded.*

Theorem 3.31 (Convergent $\Rightarrow$ Bounded) *Any convergent sequence is bounded. (This is actually originally from Lecture 12.)*

The converse is not necessarily true. E.g., take the sequence $1, -1, 1, -1, \ldots$

Theorem 3.32 (Monotonic, Then Bounded $\iff$ Convergent) *Assume $\{x_n\}$ is monotonic. Then $\{x_n\}$ converges if and only if it is bounded.*

Theorem 3.33 (Equivalent Characterization of $\limsup$) *If $\{x_n\}$ is bounded and $\limsup_{n\to\infty} x_n = M$, then*

(a) *for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n < M + \varepsilon$, and*

(b) *for all $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $k > n$ such that $x_k > M - \varepsilon$.*

Conversely, if M satisfies Properties (a) and (b), then $M = \limsup_{n\to\infty} x_n$.

Naturally, there is an analogous characterization of $\liminf$.

See 3.36 for an extension to unbounded sequences.

Theorem 3.34 (Existence of Subsequence Which Achieves $\limsup$***)*** *If* $\{x_n\}$ *is bounded and* $M = \limsup_{n\to\infty} x_n$, *then there is a subsequence of* $\{x_n\}$ *that converges to* M. *(And similarly for* $\liminf$.*) I.e., for bounded sequences, the set of subsequential limits contains the supremum and infimum. This is a corollary to* 3.33.

This implies, for example, that if our set of subsequential limits (see 3.10) looks like $\{1/n \mid n \in \mathbb{N}\}$, *then actually we must have made a mistake! Because the infimum of this set isn't included in it, and the theorem above says it should be.*

Theorem 3.35 ($\limsup$ ***of Multiplied Sequences)*** *If* $y_n \to y > 0$ *and* $\{x_n\}$ *is any sequence, then* $\limsup_{n\to\infty} x_n y_n = y \limsup_{n\to\infty} x_n$. *(Proof was given as an exercise in Lecture 14, 40:28. It is an application of 3.34.)*

Theorem 3.36 (Extension of 3.33 to Divergent Sequences) *If* $\{x_n\}$ *is a sequence and* $\limsup x_n = +\infty$, *then*

(1) there is a subsequence x_{n_k} *such that* $x_{n_k} \to +\infty$, *and*

(2) there cannot be a larger limit.

Conversely, conditions (1) and (2) also imply $\limsup x_n = +\infty$. *[This is really just because* $+\infty$ *is the only value satisfying (1) and (2).]*

Theorem 3.37 (Properties of $\limsup, \liminf$***)***

(a) $\liminf \le \limsup$ *for any sequence.*

(b) $\limsup_{n\to\infty}(-x_n) = -\limsup_{n\to\infty} x_n$

(c) $\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$ *if and only if* $\{x_n\}$ *converges, as long as you are only looking at bounded sequences. (Rudin does not say that a sequence converges to* ∞. *Rather, it diverges to* ∞.*)*

(d) $\limsup, \liminf$ *do not change when you change finitely many values in the sequence.*

(e) $\inf \operatorname{range}(\{x_n\}) \le \liminf\{x_n\} \le \limsup\{x_n\} \le \sup \operatorname{range}(\{x_n\})$, *and these inequalities may be strict. Note that* $\{x_n\}$ *is taken to be a sequence here, hence why we need to take the range before passing it to* $\inf$, *since* $\inf$ *takes sets, not sequences.*

Theorem 3.38 (Comparing $\limsup, \liminf$ ***of Two Sequences)*** *If* $x_n \le y_n$ *for all* n *greater than some* $N \in \mathbb{N}$, *then* $\limsup x_n \le \limsup y_n$ *and* $\liminf x_n \le \liminf y_n$.

Theorem 3.39 (Basic Squeeze Theorem) *If* $0 \le x_n \le y_n$ *for all sufficiently large* n *and* $y_n \to 0$, *then* $x_n \to 0$.

Theorem 3.40 (Cauchy Criterion for Series) *$\sum_{n=1}^{\infty} x_n$ converges if and only if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies*

$$\left| \sum_{k=n}^{m} x_k \right| < \varepsilon.$$

This is really just an application of 3.27 to the sequence of partial sums from our series.

Theorem 3.41 (Convergent Series Implies $\{x_n\} \to 0$) *If $\sum x_n$ converges, then $\lim_{n\to\infty} x_n = 0$. This is a quick corollary of 3.40. Just take $m = n$ in the statement of 3.40.*

The converse is clearly not true (e.g., the harmonic series).

Theorem 3.42 (Application of Monotonic to Series) *A series of nonnegative numbers converges if and only if the partial sums form a bounded sequence.*

This is just a translation of 3.32 to series. Recall that monotonic sequences converge if and only if they are bounded. Since the numbers you are summing are nonnegative, the partial sums are monotonic. Thus, they converge if and only if they are bounded.

Theorem 3.43 (Comparison Test for Convergence) *If $|a_n| \leq c_n$ for all $n \geq N$ and if $\sum_{n=1}^{\infty} c_n < \infty$, then $\sum_{n=1}^{\infty} a_n < \infty$. This is equivalent to saying: If $|a_n| \leq c_n$ for all $n \geq N$ and if the c_n's converge, then the a_n's converge.*

Theorem 3.44 (Comparison Test for Divergence) *If $a_n \geq d_n \geq 0$ for $n \geq N$ and if $\sum_{n=1}^{\infty} d_n = +\infty$, then $\sum_{n=1}^{\infty} a_n = +\infty$. (Of course, $\sum_{n=1}^{\infty} a_n = +\infty$ is equivalent to saying that $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$. See 3.9.*

Theorem 3.45 (Cauchy Sparse Sequence Theorem) *Say $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. (So we are able to show that a series of this form converges just by looking at some sparsely-placed terms.)*

Theorem 3.46 (Root Test) *Given $\sum_{n=0}^{\infty} a_n$, set*

$$\alpha = \limsup_{n\to\infty} |a_n|^{1/n}.$$

(1) If $\alpha < 1$, the series converges absolutely. (See 3.14.)

(2) If $\alpha > 1$, the series diverges.

(3) If $\alpha = 1$, we get no info.

Theorem 3.47 (Ratio Test) *Say a_n consists of nonzero terms. Consider $\sum a_n$.*

(1) If

$$\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1,$$

then $\sum a_n$ converges absolutely.

(2) If

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1,$$

then $\sum a_n$ diverges.

(3) If

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le 1 \le \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|,$$

we get no info. (E.g., $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n}$ both fall into this category. The former converges whereas the latter doesn't.)

Theorem 3.48 (Relationship Between Root/Ratio Tests) *The root test is more powerful than the ratio test, although the latter is easier to use. Formally, if the ratio test gives convergence/divergence, then the root test also gives convergence/divergence respectively. The converse is not necessarily true; see Lecture 16, 43:30. It is advised to first try the ratio test and then try the root test.*

The result follows due to the following inequalities for a sequence $\{a_n\}$ of positive numbers:

$$\liminf \frac{a_{n+1}}{a_n} \le \liminf (a_n)^{1/n},$$
$$\limsup \frac{a_{n+1}}{a_n} \ge \limsup (a_n)^{1/n}.$$

(Actually these hold for sequences of positive and negative numbers, but we only proved the positive-only version in class.)

Theorem 3.49 (Alternate Expression for e)

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Theorem 3.50 (e Is Not Rational) *e is not rational.*

Theorem 3.51 (Root Test Analog for Power Series) *Given $\sum c_n z^n$, let*

$$\alpha = \limsup_{n\to\infty} |c_n|^{1/n}.$$

Set

$$R = \begin{cases} \frac{1}{\alpha}, & 0 < \alpha < \infty \\ +\infty, & \alpha = 0 \\ 0, & \alpha = +\infty \end{cases}$$

Then the series converges for $|z| < R$ and diverges for $|z| > R$. Note that this theorem says nothing about what happens when $|z| = R$. (R is called the radius of convergence. See 3.17.)

Theorem 3.52 (Ratio Criterion for Power Series) *For the power series* $\sum c_n z^n$*, we have*

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

if the limit exists (allowing $\pm\infty$*). (Here* R *is the radius of convergence. See 3.17.)*

Theorem 3.53 (When the Root/Ratio Tests Fail to Show Convergence) *If we have* $\{a_n\}, \{b_n\}$ *such that*

(1) the partial sums of $\sum a_n$ *are a bounded sequence,*

(2) $\{b_n\}$ *is nonincreasing (i.e.,* $b_0 \geq b_1 \geq b_2 \geq \dots$*), and*

(3) $\lim_{n\to\infty} b_n = 0$*,*

then $\sum a_n b_n$ *converges.*

We can use this on certain series which the root and ratio tests fail on, such as the alternating harmonic series, $\sum \frac{(-1)^n}{n}$*. Take* $a_n = (-1)^n$ *and* $b_n = 1/n$*. Then* $\{a_n\}, \{b_n\}$ *satisfy all properties of the theorem, so the alternating harmonic series converges. (Note in particular that the partial sums of* $\sum a_n$ *are* $1, 0, 1, 0, \dots$*. Clearly this is a bounded sequence.)*

Theorem 3.54 (Sum/Distributive Properties of Convergent Series) *If* $\sum a_n = A$ *(i.e., this sum converges to* A*) and* $\sum b_n = B$ *where* $A, B \neq \pm\infty$*, then* $\sum (a_n \pm b_n) = A \pm B$*.*

Also, for all $c \in \mathbb{R}$*, we have* $\sum c a_n = cA$*.*

Theorem 3.55 (Convergence of Product of Series) *Say* $\sum_{n=0}^{\infty} a_n$ *converges absolutely and write* $\sum_{n=0}^{\infty} = A$*. Suppose we also have another convergent series* $\sum_{n=0}^{\infty} b_n = B$*. (Of course, by the definition of convergence, this means that* $A, B \neq \pm\infty$*.) Per the definition of the Cauchy product (3.18), let* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$*. Then* $\sum_{n=0}^{\infty} c_n = AB$*.*

In other words, if at least one of the two series converges absolutely, then the (Cauchy) product converges to AB*. Note that we need one of the two series to converge absolutely. Otherwise the product may actually diverge (see the example at about 23:00 in Lecture 18).*

Theorem 3.56 (Convergence of Rearrangements for Sequences) *This was not actually covered in class, but apparently rearranging a sequence does not ever change the limit—see this.*

This is why all of the theorems that follow are only for series (as it is the only interesting case).

Theorem 3.57 (Convergence of Rearrangements: Positivity) *Suppose* $\{a_n\}$ *is a sequence such that* $a_n \geq 0$ *for all* $n \in \mathbb{N}$ *and* $\{b_n\}$ *is a rearrangement of* $\{a_n\}$*. Then* $\sum a_n = \sum b_n$*.*

The $a_n \geq 0$ *is necessary. There is a counterexample in Rudin if you try to omit it.*

Theorem 3.58 (Convergence of Rearrangements: Absolute Convergence) *If* $\{a_n\}, \{b_n\}$ *are rearrangements of each other and* $\{a_n\}$ *is absolutely convergent, then so is* $\{b_n\}$*. Furthermore,* $\sum a_n = \sum b_n$*.*

This follows from 3.57.

Theorem 3.59 (Riemann's Theorem for Rearrangements) *If $\{a_n\}$ is a sequence that is convergent but not absolutely convergent, then for all $\alpha \leq \beta$ where $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$, there is a rearrangement $\{b_n\}$ of $\{a_n\}$ such that*

$$\liminf_{n\to\infty} S_n = \alpha,$$
$$\limsup_{n\to\infty} S_n = \beta,$$

where $S_n := \sum_{k=1}^{n} b_k$ is the nth partial sum of $\sum b_k$.

3.3 Proof techniques

Technique 3.60 (2ε Suffices) *If you need to show that something is less than ε, it is fine in this class to show that it is less than, e.g., 4ε. The point being that saying you can do something for any $\varepsilon > 0$ is the same as saying you can do something for any $4\varepsilon > 0$.*

Technique 3.61 (Dealing With Tail Behavior Separately) *If you have two "things you can use" and are trying to show that a sequence converges, often you will take the maximum of how far you need to go out for "thing 1" to apply with how far out you need to go for "thing 2" to apply. E.g., see the proofs of 3.26 and 3.27 in Lecture 13.*

Technique 3.62 (Proof of Convergence Via Properties) *Often you can use the existence of a convergent subsequence along with something else, like the original sequence being Cauchy, to prove the original sequence converges. See the proofs of 3.27 and 3.28 in Lecture 13.*

Technique 3.63 (Dealing With Tail Behavior Separately 2) *To bound a sequence, often you can split between tail behavior and everything before the tail. I.e., for sequence $\{x_n\}$, bound every x_n with n greater than some $N \in \mathbb{N}$ one way. The x_n with $n < N$ are then automatically bounded above since there are only finitely many. She talks about this in Lecture 14, about 30:30.*

Technique 3.64 (Proving Convergence To 0 Is Easier) *Often, if you want to show that something goes to, say, 1, it is better to show that that thing minus 1 goes to 0. It is a lot easier to show that things go to 0 than something else because if everything is positive, you just have to show that they become less than ε. E.g., in Lecture 15, we show that $\lim_{n\to\infty} n^{1/n} = 1$ by instead showing that $\lim_{n\to\infty} n^{1/n} - 1 = 0$. There is another good example in Lecture 18, 34:16.*

Technique 3.65 (Triangle Inequality and Cauchy Criteria) *The triangle inequality,*

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k|,$$

is very useful in conjunction with the Cauchy criteria for series, 3.40. See Lecture 15, 39:51 for an application.

Technique 3.66 (Bounding Away) *Often if you have an inequality $x < 1$, it maybe be useful to write $x < r$, where $r < 1$. For an application, see 31:55 in Lecture 16. (Often this is useful when you want to compare something to the geometric series to show that it converges.)*

Technique 3.67 (Bounding Away and Taming the Tail) *The "Proof of 2nd Inequality" done at the end of Lecture 16 is very instructive. It uses two common tricks. The first is very similar to the tip above where you create this r so that you can say that x is strictly less than something strictly less than 1. Here, we set $\alpha \leftarrow \limsup \frac{a_{n+1}}{a_n}$. We then pick some $\beta > \alpha$ and use the fact that there exists $N \geq 1$ such that $n \geq N$ implies $\frac{a_{n+1}}{a_n} \leq \beta$ in our proof. (Furthermore, the fact that this holds for all $\beta > \alpha$ is critical in getting our final upper bound.)*

The above gives us that $a_{N+k+1} \leq \beta a_{N+k}$ for all $k \geq 1$. By induction, we have

$$a_{N+k} \leq \beta^k a_N$$

for all $k \geq 1$. This is the second trick: introducing this fixed a_N which then allows us to get an upper bound on the tail behavior.

Technique 3.68 (Prove That a Sequence Converges to Specific Value Using $\limsup, \liminf$) *At the beginning of Lecture 17, we prove 3.49. Instead of directly showing that the limit is e using the definition of convergence, we instead use a different technique. We show that*

$$\limsup \left(1 + \frac{1}{n}\right)^n \leq e,$$
$$\liminf \left(1 + \frac{1}{n}\right)^n \geq e,$$

and then apply 3.37 Part (e), i.e., the $\liminf$ is always at most the $\limsup$. Then

$$\limsup \left(1 + \frac{1}{n}\right)^n = \liminf \left(1 + \frac{1}{n}\right)^n = e,$$

and so the sequence converges to e.

We also use a neat little technique to get the initial lower bound on the $\liminf$ by introducing a new variable, m, which allows us to lower bound the nth term in the sequence when $n \geq m$. We then take the limit as $n \to \infty$ of the expression with the m in it. Finally, we remove the m by taking $m \to \infty$, which makes since because as $n \geq m$, doing this "drives" n to ∞. (The first time we take $n \to \infty$, we are just looking at what happens in our lower bound when $n \to \infty$. The second time we actually look at the tail behavior by considering elements farther and farther down in the sequence. Specifically, we look at the nth element as $n \to \infty$.)

It would be worth going over this trick more formally when reviewing. She explains it very clearly at about 17:40 in Lecture 17.

Technique 3.69 (Change of Variables When Applying Ratio Criterion for Power Series) *See 3.52 for the ratio criterion for power series. At about 44:00 in Lecture 17, we show how to apply this if everything isn't centered at 0. In particular, consider the power series*

$$1 + (z-1) + (z-1)^2 + \cdots = \sum (z-1)^n.$$

We can do a change of variables $y \leftarrow z - 1$ so that this becomes $\sum y^n$. The ratio criterion for power series gives that this converges when $|y| < 1$. So the original series converges when $|z - 1| < 1$.

Technique 3.70 (Using $\limsup$ **When You Don't Know Whether It Converges)** *In the proof of 3.55, we want to show that* $\sum_{n=0}^{\infty} c_n = AB$. *We do this by showing that*

$$\limsup_{n\to\infty} |C_n - A_n B| \le \varepsilon\alpha.$$

Why do it this way? Well, we know for a fact that the $\limsup$ *exists because it always does. So if we can bound the* $\limsup$ *of the magnitude toward 0, we automatically know that the* $\lim$ *exists and it is 0 as well. (Formally, we want to argue that the* $\liminf$ *must also be 0 so that* $\lim = \liminf = \limsup = 0$.*) But the point is that we do have to use the* $\limsup$ *initially. We aren't allowed to write* $\lim$ *until we are sure it converges.*

Technique 3.71 (The Original Sequence is a Rearrangement of the Rearrangement) *In the proof of 3.57 (Lecture 18, very end), there is a nice little technique that makes the proof super short. If you are trying to prove that* $\sum a_n = \sum b_n$ *where* $\{b_n\}$ *is a rearrangement of* $\{a_n\}$, *you only actually need to show that* $\sum a_n \le \sum b_n$ *(or the flipped version). This is because* $\{a_n\}$ *is also a rearrangement of* $\{b_n\}$, *so you can always get the other inequality by relabeling.*

4 Continuity

4.1 Definitions

Definition 4.1 (Limit 1) *Let* X, Y *be metric spaces. Suppose* $E \subset X$, *and* $f : E \to Y$. *If* p *is a limit point of* E *(note that* p *need not be in* E*), we say* $\lim_{x\to p} f(x) = q$ *if* $q \in Y$ *satisfies: for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $0 < d_X(x,p) < \delta$ *and* $x \in E$ *implies* $d_Y(f(x), q) < \varepsilon$.

Note that since p *need not be in* E, $f(p)$ *need not be defined, but the limit might exist anyway. Even if* $f(p)$ *is defined, its value is not relevant to the limit.*

See 4.2 for an equivalent definition.

If $\lim_{x\to p} f(x) = q$, *then* q *is unique. (The limit is unique if it exists.)*

Definition 4.2 (Limit 2) *Define* X, Y, E, F, p *as in 4.1, our first definition of the limit. Then* $\lim_{x\to p} f(x) = q$ *if and only if* $\lim_{n\to\infty} f(p_n) = q$ *for all sequences* $\{p_n\}$ *in* E *such that* $p_n \ne p$ *for all* n *and* $\lim_{n\to\infty} p_n = p$.

Note that the $p_n \ne p$ *condition is necessary for the definitions to be equivalent. It is true that without the* $p_n \ne p$ *condition, this definition still implies our first definition. However, the converse is not true. So we do need* $p_n \ne p$.

Definition 4.3 (Continuous/Continuity) *Let* X, Y *be metric spaces with* $E \subset X$ *and* $f : E \to Y$. *Then* f *is continuous at* $p \in E$ *if for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x \in E$ *with* $d_X(x,p) < \delta$, *we have* $d_Y(f(x), f(p)) < \varepsilon$.

The difference between this and the definition of the limit is that we need $f(p)$ *to be defined (and take the right value.) In particular, if* $\lim_{x\to p} f(x)$ *exists, then* f *is continuous at* p *if and only if* $f(p) = \lim_{x\to p} f(x)$.

If f *is continuous for all* $p \in E$, *then we say that* f *is continuous on* E.

As a special case, if p is isolated in X and $f(p)$ is defined, then f is continuous at p. (This just follows from the definition even though it seems counterintuitive.)

Definition 4.4 (Uniformly Continuous/Uniform Continuity) *Let $f : X \to Y$ where X, Y are metric spaces. Then f is uniformly continuous on X if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$ with $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$.*

Note: This does not make sense at a point, only on a set. Furthermore, it implies continuity, meaning that if $f : X \to Y$ is uniformly continuous, then f is continuous at all $x \in X$.

The difference between this and regular continuity is that we can pick the same δ (for a given $\varepsilon > 0$) at any point to show that the function is continuous at that point. For regular continuity, δ might depend on the point you are looking at. Uniform continuity is saying that it can be independent of the point you are looking at. (Of course, δ will still depend on ε though.)

Definition 4.5 (Linear, Piecewise Linear) *See end of Lecture 22.*

Definition 4.6 (One-Sided Limits) *Recall that $f(x^+)$ is the limit where you approach from the right. (You start from $+\infty$ and move to the left.) $f(x^-)$ is the limit where you approach from the left. (So you start from $-\infty$ and move to the right.)*

Definition 4.7 (Types of Discontinuities) *If f is discontinuous at x and $f(x^+), f(x^-)$ exist, then x is a discontinuity of the 1st kind. (Also called a simple discontinuity.)*

Otherwise, x is a discontinuity of the 2nd kind.

Definition 4.8 (Monotonic Functions) *$f : (a, b) \to \mathbb{R}$ is monotonically increasing if $a < x < y < b$ implies $f(x) \leq f(y)$. A monotonically decreasing function is defined similarly. f is monotonic if it is either monotonically increasing or monotonically decreasing.*

Definition 4.9 (Extended Limits) *A way of making sense of limits at infinity. See the end of Lecture 24.*

All the usual properties of the limit hold, although we have to be a bit careful.

4.2 Theorems

Theorem 4.10 (Properties of the Limit) *Let $\lim_{x\to p} f(x) = q$ and $\lim_{x\to p} g(x) = q'$. Then*

(a) $\lim_{x\to p}(f \pm g)(x) = q \pm q'$,

(b) $\lim_{x\to p}(fg)(x) = qq'$, *and*

(c) $\lim_{x\to p}(f/g)(x) = q/q'$ *when* $q' \neq 0$.

The easiest way to prove these is to use properties of sequences we have already derived in conjunction with our definition of the limit in terms of sequences, 4.2.

Theorem 4.11 (Continuity and Open/Closed Sets) *Let X, Y be metric spaces with $f : X \to Y$. Then f is continuous if and only if $f^{-1}(V)$ is open in X for all open $V \subset Y$. Equivalently, f is continuous if and only if $f^{-1}(F)$ is closed in X for all closed $F \subset Y$. (This theorem was phrased in terms of the entire metric space X, but it can also be phrased in terms of some subset E of X.)*

Note that $f^{-1}(V)$ is the inverse image of V. We are not saying that f is invertible. Note also that the image of f need not be all of Y. If you take some open/closed set $V \subset Y$ which contains no points in the image of f, then $f^{-1}(V) = \emptyset$. $\emptyset$ is closed and open, so the theorem still holds.

Intuitively, we can think about this theorem as saying that continuity is equivalent to f "pulling open sets back to open sets" (and similarly for closed sets). A natural question then is whether continuity also implies that open sets are pushed to open sets. This isn't the case however. Pick f to be a constant function, i.e., a function which maps everything in X to a single point in Y. Take some open set $V \subset X$. $f(V)$ is a single point, so assuming that Y doesn't have a discrete metric, $f(V)$ isn't open.

Theorem 4.12 (Continuity Is Preserved Under Composition) *If $f : X \to Y, g : Y \to Z$, f is continuous at x, and g is continuous at $f(x)$, then $g \circ f$ is continuous at x. This follows almost immediately from 4.11.*

Theorem 4.13 (Continuity Is Preserved Under Arithmetic Operations) *If f, g are functions with values in $\mathbb{R}^k$, then if f, g are continuous, we have that $f \pm g, fg, f/g$ are all continuous. (For the last one, we also need $g \neq 0$).*

The easiest way to prove this is to use properties of the limits of sequences which we have already derived in conjunction with our second definition of the limit, 4.2.

Theorem 4.14 (Continuity of Vector-Valued Functions) *If $\overline{f} : x \to \mathbb{R}^k$ is a vector-valued function, i.e., $\overline{f}(x) = (f_1(x), \ldots, f_k(x))$ where each f_i is called a component, then f is continuous if and only if each f_i is continuous.*

Note that this is not the definition of continuity for a vector-valued function; $\mathbb{R}^k$ has its own metric (the Euclidean norm) and the original definition is still the definition here (call $\mathbb{R}^k$ Y, etc.). This is just an equivalent characterization when we have this special property that our function is vector-valued.

Theorem 4.15 (Compactness Is Preserved Under Continuous Functions) *Let X, Y be metric spaces where X is compact (but not necessarily Y). Then if $f : X \to Y$ is continuous, $f(X)$ is compact.*

Intuitively, what this means is that continuous functions "push forward" compactness. Note that continuous functions do not push forward the properties of being open or closed. As we saw in 4.11, they only "pull back" the property of being closed/open.

As an immediate corollary of this result, we have that if $f : X \to Y$ is continuous and X is a compact metric, then $f(x)$ is closed and bounded. (Compact sets are always closed and bounded, although the converse isn't true in general metric spaces.)

Theorem 4.16 (Continuity, Compactness, $\mathbb{R}$, and Achieving $\inf, \sup$) *Let $f : X \to \mathbb{R}$ be* *continuous and X be a compact metric space. Set*

$$M = \sup_{x \in X} f(x),$$
$$m = \inf_{x \in X} f(x).$$

Then there exists points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

In other words, a continuous, real-valued function on a compact domain achieves its infimum and supremum.

The theorem follows because $f(X) \subset \mathbb{R}$ is compact, which implies that it is closed and bounded. And we already know that closed and bounded subsets of $\mathbb{R}$ achieve their supremum and infimum.

Theorem 4.17 (Compactness, Continuity Imply Inverse Is Continuous) *If f is continuous, bijective map from a compact metric space X onto Y (so the image is all of Y), then f^{-1} is a continuous (bijective) mapping of Y onto X. (The inverse is defined as the function f^{-1} such that $f^{-1}(f(x)) = x$ for all $x \in X$.)*

We do need our map to be bijective and not just surjective. (I.e., the image must be all of Y.) See the proof given in the lecture for details.

Theorem 4.18 (Continuous Map From Compact Implies Uniformly Continuous) *If $f : X \to Y$ is continuous, X is a compact metric space, and Y is a metric space, then f is uniformly continuous.*

Theorem 4.19 (Continuous, Connected) *If $f : X \to Y$ is continuous and if $E \subset X$ is connected, then $f(E)$ is connected.*

Theorem 4.20 (Intermediate Value Theorem (IVT)) *If $f : [a, b] \to \mathbb{R}$ is continuous, $f(a) < f(b)$, and $c \in (f(a), f(b))$, then there exists $x \in (a, b)$ such that $f(x) = c$.*

The converse is not true.

This is a corollary of the previous theorem (Continuous, Connected).

Theorem 4.21 (Approximating Using Piecewise Linear) *If $f : I \to \mathbb{R}$ is continuous and I is a closed, bounded interval, then for all $\varepsilon > 0$, there exists a piecewise linear function $g : I \to \mathbb{R}$ such that $|f(t) - g(t)| < \varepsilon$ for all $t \in I$.*

Theorem 4.22 (Monotonic Functions One-Sided Limit Bounds) *If f is monotonically increasing on (a, b), then $f(x^+), f(x^-)$ exist for all $x \in (a, b)$. And for each $x \in (a, b)$, we have*

$$\sup_{a<t<x} f(t) = f(x^-) \leq f(X) \leq f(x^+) = \inf_{x<t<b} f(t).$$

As a corollary, for $a < x < y < b$, we have $f(x^+) \leq f(y^-)$.

Theorem 4.23 (Monotonic Functions Can't Have 2nd-Type Discontinuities) *Monotonic functions can only have discontinuities of the 1st type. This is a corollary of the theorem above (Monotonic Functions One-Sided Limit Bounds).*

Theorem 4.24 (Monotonic Implies Countably Many Discontinuities) *If $f : (a, b) \to \mathbb{R}$ is monotonic, then it can only have countably many discontinuities.*

5 Differentiation

5.1 Definitions

Definition 5.1 (Differentiable) *Say $f : [a, b] \to \mathbb{R}$. f is differentiable at $x \in [a, b]$ if*

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

exists. (We take the one-sided limit if x is an endpoint). We write the value as $f'(x)$.

Definition 5.2 (Local Maximum/Minimum) *Suppose $f : [a, b] \to \mathbb{R}$. f has a local maximum at $x \in [a, b]$ if there exists a neighborhood U of c such that $f(x) \geq f(x)$ for all $x \in U \cap [a, b]$. We define a local minimum analogously.*

Definition 5.3 (n-th Derivative) *See Lecture 27. We use the notation $f^{(n)}$ for the n-th derivative.*

Definition 5.4 (Derivative of $f : \mathbb{R} \to \mathbb{R}^k$) *See Lecture 28.*

It is still true that differentiability at x for such a function f implies that f is continuous at x. However, the ordinary MVT fails. To generalize it, we must turn the equality into an inequality.

5.2 Theorems

Theorem 5.5 (Differentiable Implies Continuous) *If $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b]$, then f is continuous at x. The converse is not true.*

Theorem 5.6 (Product/Quotient Rules) *If $f, g : [a, b] \to \mathbb{R}$ are differentiable at some $x \in [a, b]$, then so are $f \pm g, fg, f/g$, and*

$$\begin{aligned}
(f \pm g)'(x) &= f(x) \pm g'(x) \\
(fg)'(x) &= f(x)g'(x) + f'(x)g(x) \\
(f/g)'(x) &= \frac{g(x)f'(x) + f(x)g'(x)}{(g(x))^2}.
\end{aligned}$$

(In the last case, we need to make sure that $g(x) \neq 0$.)

Theorem 5.7 (Chain Rule) *Suppose f is continuous on $[a, b]$ and that $f'(x)$ exists for some $x \in [a, b]$. If g is a function defined on some interval containing $f(I)$ where $x \in I$, and g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x and*

$$(g \circ f)'(x) = g'(f(x))f'(x).$$

Theorem 5.8 (Derivative Vanishes at Local Max/Min) *If $f : [a, b] \to \mathbb{R}$ has a local maximum or minimum at $c \in (a, b)$ and if $f'(x)$ exists, then $f'(c) = 0$.*

Theorem 5.9 (Mean Value Theorem (MVT)) *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Interpretation: You achieve the average speed at some particular time.

This is a corollary of the next theorem (Generalized Mean Value Theorem). (Take $g(x) = x$.)

Theorem 5.10 (Generalized Mean Value Theorem (Generalized MVT)) *If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then there exists $x \in (a, b)$ such that*

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

Theorem 5.11 (Using Derivative to Show Monotonicity) *If f is differentiable on (a, b), then*

- *$f'(x) \geq 0$ for all $x \in (a, b)$ implies that f is monotonically increasing,*
- *$f'(x) \leq 0$ for all $x \in (a, b)$ implies that f is monotonically decreasing, and*
- *$f'(x) = 0$ for all $x \in (a, b)$ implies that f is constant.*

This is a corollary of MVT.

Theorem 5.12 (Intermediate Value Theorem (IVT) for Derivatives) *If $f : [a, b] \to \mathbb{R}$ is differentiable for all $x \in [a, b]$ and if $f'(a) < y < f'(b)$, then there exists $x \in (a, b)$ such that $f'(x) = y$. (Analogous statement for $f'(a) > y > f'(b)$.)*

Note that this holds even if the derivative is discontinuous (which can happen even if it exists everywhere).

Theorem 5.13 (Derivative Can't Have 1st-Type Discontinuities) *If $f : [a, b] \to \mathbb{R}$ is differentiable, then f' cannot have any discontinuities of the first kind. This is a corollary of the previous theorem (IVT for Derivatives).*

As a corollary of this theorem, it follows that not all functions can be the derivative of some function.

Theorem 5.14 (L'Hospital's Rule) *Suppose f, g are real and differentiable in (a, b), where $-\infty \leq a < b \leq +\infty$. (Note that a, b can be $\pm\infty$.) Suppose*

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0.$$

Then if $g' \neq 0$ on (a, b) and the limit

$$L = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

exists, then g never vanishes on (a, b) (i.e., $g \neq 0$ on (a, b)) and

$$L = \lim_{x \to a^+} \frac{f(x)}{g(x)}.$$

The same result holds for the left limit with the right endpoint, any 2-sided limit in (a, b), and a limit that goes to $\pm\infty$. (We covered the last one already since we remarked that a, b can be $\pm\infty$.)

Theorem 5.15 (Taylor's Theorem) *Assume f is n-times differentiable on $[a, b]$, and $f^{(n)}$ is continuous on $[a, b]$ and differentiable on (a, b). Let $x_0 \in [a, b]$. Then for all $x \in [a, b]$ with $x \neq x_0$, there exists $c \in (x, x_0)$ such that*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Note that this is a generalization of the MVT. Take $x = b, x_0 = a, n = 1$; this gives $f(b) = f(a) + f'(c)(b - a)$, which is just the MVT.

Theorem 5.16 (Version of MVT for $f : \mathbb{R} \to \mathbb{R}^k$) *See Lecture 28. To generalize it, you have to turn the equality into an inequality (and take magnitudes).*